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Statistical Λ $-$Convergence of Order α in Intuitionistic Fuzzy Normed Spaces

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ABSTRACT: In this paper, we have used Λ-convergence introduced by Mursaleen [19]. Let $\lambda = (\lambda_n)$ be a real sequence of positive numbers that approaches to infinity with $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$ which we have used for introducing statistical Λ-convergence. We have developed a generalized characterization named as statistical Λ-convergence of order α $(0 < \alpha \le 1)$ in the intuitionistic fuzzy normed space (IFNS) and obtained some results with suitable examples.

Keywords: Statistical convergence, Statistical Λ-convergence, Intuitionistic fuzzy normed space

I. INTRODUCTION

Firstly, Zygmund [25] proposed the idea of statistical convergence in 1935. Fast [8] simultaneously brought an expansion to the normal concept of sequence convergence which he named as Statistical convergence. Schoenberg [21] and Fridy [9] provided a few fundamental properties of this convergence and also explained this result by summability method independently. Over a period of time plenty of new research work has been taken place in this area because of its applicability in the various mathematical fields as in Banach spaces [10], Fourier analysis [1], measure theory [4, 15], number theory [16] and trigonometric series [25] etc. Earlier this concept was restricted only to the sequences of real and complex numbers but later on it has been extended to sequences in probabilistic normed space[12], random normed space [5], fuzzy normed space [23] and intuitionistic fuzzy normed space [17].

In brief outline, the formulation of statistical convergence is intimately connected to natural density. The natural density of any subset E of $\mathbb{N}$ (set of natural numbers) can be approximated by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{r \in E : r \le n\}|$$

where, $|.|$ represents the cardinality of the enclosed set E.

Definition 1.1 [8] A sequence $y = (y_r)$ is statistically convergent to some l if for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \tfrac{1}{n} |\{r \le n : |y_r - l| > \epsilon\}| = 0 \quad (1.1)$$

Since statistical convergence is of great importance therefore, a lot of research work has been done on its generalization. One of which is λ $-$statistical convergence which was given by Mursaleen [18] using the non-decreasing sequence $\lambda = (\lambda_r)$ approaches to ∞ in a way that $\lambda_{r+1} \le \lambda_r + 1,\ \lambda_1 = 1$.

Definition 1.2 [18] A sequence $y = (y_r)$ converges λ-statistically convergent to some l if for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \tfrac{1}{\lambda_n} |\{r \in I_n : |y_r - l| \ge \epsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$.

Therefore, it could be written as $S_\lambda - \lim_{r\to\infty} y_r = l$ or $y_r \to l(S_\lambda)$ and $S_\lambda = \{y : l \in \mathbb{R}, S_\lambda - \lim y = l.\}$

For any $\alpha \in (0,1]$ Çolak along with Bektaș [6] in their paper, they generalized the above definition and named it as λ-statistical convergence of order α.

Definition 1.3 [6] Let $\alpha \in (0,1]$ and $\lambda = (\lambda_r)$ be an increasing sequence of positive real numbers. Then, a sequence $y = (y_r)$ converges λ-statistically of order α $(0 < \alpha \le 1)$ to some l if for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{r \in I_n : |y_r - l| \ge \epsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$.

Therefore, it could be written as $S_\lambda^\alpha - \lim_{r\to\infty} y_r = l$ or $y_r \to l(S_\lambda^\alpha)$ and $S_\lambda^\alpha = \{y : l \in \mathbb{R}, S_\lambda^\alpha - \lim y = l\}$.

In this paper we will work on statistical Λ-convergence of order α in the intuitionistic fuzzy normed space (IFNS). Firstly, we are discussing about a few necessary concepts related to IFNS. The concept of fuzziness of a set was proposed in the year 1965 by Zadeh [24]. His notion brought a drastic change in the concept of classical set theory. As it is a tool of much wider applicability in those conditions where the norm of any vector is not possible to evaluate. So, with the passing years, research work on this topic has been increased on a rapid scale because of its use in the various fields as in population control [3], fuzzy topology [7], non-linear dynamics system [11] etc. Later on, Park [14] in 2004 came up with fuzzy intuitionistic metric space and then in the year 2006 along with Saadati [20] he proposed the IFNS. Next, we are mentioning the basic terms used in the concept of IFNS.

Definition 1.4 [22] A t-norm which is also known as triangular norm is defined as a binary operation $*$ on $[0,1]$ as continuous mapping $* : [0,1] \times [0,1] \to [0,\ 1]$ $\forall\, p, q, r, s \in [0,1]$ if the properties given below are satisfied:

a) $p * 1 = p,$
b) $p * q = q * p,$
c) $(p * q) * r = p * (q * r),$
d) $p * q \le r * s$ if $r \ge p$ and $s \ge q.$

We will move forward to the next definition that is useful for understanding and deriving IFNS.

Definition 1.5 [22] A t-conorm is defined as a binary operation $\circ$ on $[0,1]$ as a continuous mapping $\circ: [0,1] \times [0,1] \to [0,1]$ $\forall\, p,q,r,s \in [0,1]$ if the properties given below are satisfied:

a) $p \circ 0 = p$,
b) $p \circ q = q \circ p$,
c) $(p \circ q) \circ r = p \circ (q \circ r)$,
d) $p \circ q \le r \circ s$ if $r \ge p$ and $s \ge q$.

Using definition 1.4 and 1.5 Park along with Saadati [20], proposed the structure of IFNS as follows:

Definition 1.6 [14]: A 5-tuple $(Y, \varphi, \vartheta, *, \circ)$ is known as intuitionistic fuzzy normed space (IFNS) if Y is a vector space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm and (φ, ϑ) are fuzzy sets on $Y \times (0,\infty)$ satisfies the next properties for any $x, y \in Y$ and $p,\ q > 0$:

a) $\varphi(x,p) + \vartheta(x,p) \le 1$,
b) $\varphi(x,p) > 0$ and $\vartheta(x,p) < 1$,
c) $\varphi(x,p) = 1$ and $\vartheta(x,p) = 0$ iff $x = 0$,
d) $\varphi(\alpha x,p) = \varphi\left(x, \frac{p}{|\alpha|}\right)$ and $\vartheta(\alpha x,p) = \vartheta\left(x, \frac{p}{|\alpha|}\right)$ for each $\alpha \ne 0$,
e) $\varphi(x,p) * \varphi(y,q) \le \varphi(x+y, p+q)$ and $\vartheta(x,p) \circ \vartheta(y,q) \ge \vartheta(x+y, p+q)$,
f) $\varphi(x,\circ): (0,\infty) \to [0,1]$ and $\vartheta(x,\circ): (0,\infty) \to [0,1]$ are continuous,
g) $\lim_{p\to\infty} \varphi(x,p) = 1$, $\lim_{p\to 0} \varphi(x,p) = 0$, $\lim_{p\to\infty} \vartheta(x,p) = 0$ and $\lim_{p\to 0} \vartheta(x,p) = 1$.

Then the set (φ, ϑ) is named as intuitionistic fuzzy norm. This definition can be explained with the help of an example as given below:

Example Let $(Y, \|\cdot\|)$ be a normed space. Consider $m * n = \min\{m+n, 1\}$ and $m * n = mn$ on $[0,1]$ $\forall$ y $\in Y$ and every $p > 0$, take

$$\varphi(y,p) = \frac{p}{p+\|y\|} \text{ and } \vartheta(y,p) = \frac{\|y\|}{p+\|y\|}.$$

Then, $(Y, \varphi, \vartheta, *, \circ)$ is an IFNS.

Sadaati and Park [20] studied the notion of convergence and Cauchy sequence in IFNS which is given below:

Definition 1.7 [20] Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ converges to some $l \in Y$ with respect to (φ, ϑ) if for each $\epsilon > 0$ and $p > 0$ $\exists\, r_0 \in \mathbb{N}$ such that $\varphi(y_r - l, p) > 1 - \epsilon$ and $\vartheta(y_r - l, p) < \epsilon$ $\forall\, r \ge r_0$. Symbolically, $(\varphi, \vartheta) - \lim_{r\to\infty} y_r = l$.

Definition 1.8 [13] Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ converges statistically to some $l \in Y$ with respect to (φ, ϑ) if for each $\epsilon > 0$ and $p > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{r \in \mathbb{N} : \varphi(y_r - l, p) \le 1 - \epsilon \; or \; \vartheta(y_r - l, p) \ge \epsilon\}| = 0,$$

or

$$\lim_{n\to\infty} \frac{1}{n} |\{r \in \mathbb{N} : \varphi(y_r - l, p) > 1 - \epsilon \text{ and } \vartheta(y_r - l, p) < \epsilon\}| = 1.$$

II. MAIN RESULTS

In order to study statistical Λ-convergence in IFNS for order α, we first mention Λ-convergence of the sequences given by Mursaleen [19] in the year 2010 as follows:

A sequence $y = (y_r)_{r=1}^{\infty}$ is Λ-convergent to $l \in Y$ if $\Lambda y_r \to l$ as $r \to \infty$ where $\Lambda y_r = \frac{1}{\lambda_r} \sum_{j=0}^{r} (\lambda_j - \lambda_{j-1}) y_j$, and the sequence (λ_j) with $0 < \lambda_0 < \lambda_1 < \cdots \ldots\ldots < \lambda_j < \cdots \ldots$ and $\lambda_j \to \infty$ as $j \to \infty$.

Definition 2.1 [2] Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ is said to be Λ-convergent to some $l \in Y$ with respect to (φ, v) if for every $\epsilon > 0$ and $p > 0$ $\exists\, r_0 \in \mathbb{N}$ such that $\varphi(\Lambda y_r - l, p) > 1 - \epsilon$ and $\vartheta(\Lambda y_r - l, p) < \epsilon$, $r \ge r_0$.

Definition 2.2 [2] Let $(Y, \varphi, v, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ is said to be statistically Λ-convergent to some $l \in Y$ with respect to (φ, ϑ) if for every $\epsilon > 0$ and $p > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) \le 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - l, p) \ge \epsilon\}| = 0,$$

or

$$\lim_{n\to\infty} \frac{1}{n} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - l, p) < \epsilon\}| = 1.$$

It could be presented as $S_{\Lambda}^{\varphi,\vartheta} - \lim_{r\to\infty}(y_r) = l$ where $S_{\Lambda}^{\varphi,\vartheta}(Y)$ represents the collection of all statistically Λ-convergent sequences in Y.

Definition 2.3 Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ is said to be statistically Λ–convergent of order $\alpha (0 < \alpha \le 1)$ to $l \in Y$ with respect to (φ, ϑ) if for every $\epsilon > 0$ and $p > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - l, p) < \epsilon\}| = 1$$

or

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) \le 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - l, p) \ge \epsilon\}| = 0$$

And could be written as $S_{\Lambda}^{\varphi,\vartheta^{\alpha}} - \lim_{r\to\infty}(y_r) = l$ where $S_{\Lambda}^{\varphi,\vartheta^{\alpha}}(Y)$ represents the collection of all the sequences in Y which are Λ-statistically convergent of order α in the IFNS.

In case when $\alpha = 1$ then definition 2.3 gives same result as that of definition 2.2 since they both coincide.

Definition 2.4 Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ is said to be statistically Λ-Cauchy of order α $(0 < \alpha \le 1)$ with respect to (φ, ϑ) if for every $\epsilon > 0$ and $p > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_s, p) \le 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \Lambda y_s, p) \ge \epsilon\}| = 0,$$

or

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_s, p) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - \Lambda y_s, p) < \epsilon\}| = 1.$$

Lemma 2.5 Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS and $\lambda = (\lambda_r)_{r=1}^{\infty}$ be a real non-decreasing sequence of positive numbers defined above.

The next statements are equivalent for the sequence $y = (y_r)_{r=1}^{\infty}$ whenever $\epsilon > 0$ and $p > 0$.

(i) $S_{\Lambda}^{\varphi,\vartheta^{\alpha}} - \lim_{r\to\infty}(y_r) = l$,

(ii) $\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) \le 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - l, p) \ge \epsilon\}| = 0$

(iii) $\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - l, p) < \epsilon\}| = 1$,

(iv) $S_{\Lambda}^{\varphi,\vartheta^{\alpha}} - \lim \varphi(\Lambda y_r - l) = 1$ and $S_{\Lambda}^{\varphi,v^{\alpha}} - \lim \vartheta(\Lambda y_r - l) = 0$.

Now, we will prove that the uniqueness property of the limit of sequence in IFNS for the statistical Λ-convergence.

Theorem 2.6 Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS. A sequence $y = (y_r)_{r=1}^{\infty}$ is statistically Λ-convergent of order $\alpha (0 < \alpha \le 1)$ with respect to (φ, ϑ) to unique limit.

Proof: Let $S_{\Lambda}^{\varphi,\vartheta^{\alpha}} - \lim_{r\to\infty}(y_r) = l_1$ and $S_{\Lambda}^{\varphi,\vartheta^{\alpha}} - \lim_{r\to\infty}(y_r) = l_2$ such that $l_1 \ne l_2$.

For any $\epsilon > 0$, we choose $w > 0$ such that $(1-w) * (1-w) > 1 - \epsilon$ and $w \circ w < \epsilon$.

Now, for $p > 0$, define

$$L_{\varphi,1}(w,p)=\left\{r\in\mathbb{N}:\varphi\left(\Lambda y_r-l_1,\frac{p}{2}\right)\le 1-w\right\},$$
$$L_{\varphi,2}(w,p)=\left\{r\in\mathbb{N}:\varphi\left(\Lambda y_r-l_2,\frac{p}{2}\right)\le 1-w\right\},$$
$$L_{\vartheta,1}(w,p)=\{r\in\mathbb{N}:\vartheta\left(\Lambda y_r-l_1,\frac{p}{2}\right)\ge w\},$$
$$L_{\vartheta,2}(w,p)=\left\{r\in\mathbb{N}:\vartheta\left(\Lambda y_r-l_2,\frac{p}{2}\right)\ge w\right\}.$$

Since, $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(y_r)=l_1$ and $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(y_r)=l_2$.

Then it follows for every $p>0$,

$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\varphi,1}(w,p)\right|=0$ and $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\varphi,2}(w,p)\right|=0$

$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\vartheta,1}(w,p)\right|=0$ and $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\vartheta,2}(w,p)\right|=0$

Let

$L_{\varphi,\vartheta}(w,p)=\{L_{\varphi,1}(w,p)\cup L_{\vartheta,1}(w,p)\}\cap\{L_{\varphi,2}(w,p)\cup L_{\vartheta,2}(w,p)\}$.

Then clearly $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\varphi,\vartheta}(w,p)\right|=0$

$\Rightarrow\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\varphi,\vartheta}^c(w,p)\right|=1$.

If $r\in L_{\varphi,\vartheta}^c(w,p)$, then we have two cases.

$$r\in\{L_{\varphi,1}(w,p)\cup L_{\varphi,2}(w,p)\}^c,$$
$$r\in\{L_{\vartheta,1}(w,p)\cup L_{\vartheta,2}(w,p)\}^c.$$

First, we consider

If $r\in\{L_{\varphi,1}(w,p)\cup L_{\varphi,2}(w,p)\}^c$. Then, we have

$$\varphi(l_1-l_2,p)\ge\varphi\left(\Lambda y_r-l_1,\frac{p}{2}\right)*\varphi\left(\Lambda y_r-l_2,\frac{p}{2}\right)$$
$$>(1-w)*(1-w)$$
$$>1-\epsilon$$

$\Rightarrow\varphi(l_1-l_2,p)>1-\epsilon$

Since, $\epsilon>0$ was arbitrary, then $\varphi(l_1-l_2,p)=1$ for all $p>0\Rightarrow l_1=l_2$.

(b) If $r\in\{L_{\vartheta,1}(w,p)\cup L_{\vartheta,2}(w,p)\}^c$. Then, we have

$$\vartheta(l_1-l_2,t)\le\vartheta\left(\Lambda y_r-l_1,\frac{p}{2}\right)\circ\vartheta\left(\Lambda y_r-l_2,\frac{p}{2}\right)<w\circ w<\epsilon$$

$\Rightarrow\vartheta(l_1-l_2,p)<\epsilon$

$\Rightarrow\vartheta(l_1-l_2,p)=0$ for $p>0$.

$\Rightarrow l_1=l_2$.

Hence, we get unique limit.

Next, we are giving the relation of usual convergence and statistical Λ –convergence of order α.

Theorem 2.7 Let $(Y,\varphi,\vartheta,*,\circ)$ be an IFNS and $y=(y_r)_{r=1}^\infty$ be any sequence. If $(\varphi,\vartheta)-\lim_{r\to\infty}(y_r)=l$, then $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(y_r)=l$ where $\alpha\in(0,1]$. But, converse does not hold.

Proof: Since $(\varphi,\vartheta)-\lim y_r=l$, then for any $\epsilon>0$ and $p>0$ there exists a number $r_0\in\mathbb{N}$ such that $\varphi(\Lambda y_r-l,p)>1-\epsilon$ and $\vartheta(\Lambda y_r-l,p)<\epsilon\ \ \forall\, r\ge r_0$.

$L_{\varphi,\vartheta}=\{r\in\mathbb{N}:\varphi(\Lambda y_r-l,p)\le 1-\epsilon \text{ or } \vartheta(\Lambda y_r-l,p)\ge\epsilon\}$ contains finite number of elements. As every finite subset of natural numbers $\mathbb{N}$ has density equals to zero.

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^\alpha}|\{r\in\mathbb{N}:\varphi(\Lambda y_r-l,p)\le 1-\epsilon \text{ or } \vartheta(\Lambda y_r-l,p)\ge\epsilon\}|=0.$$

i.e. $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(y_r)=l$.

But converse part does not holds, it can be explained with the help of next example.

Theorem 2.8 Let $(\mathbb{R},|\cdot|)$ be real normed space under usual norm.

Define, $c*f=cf$ and $c\circ f=\min(1,c+f)$

Let $\varphi(x,p)=\frac{p}{p+|x|}$ and $\vartheta(x,p)=\frac{|x|}{p+|x|}\ \forall\, x\in\mathbb{R}$ and $p>0$

Define $\Lambda y_r=\begin{cases}1, & r=m^2\\ 0, & \text{otherwise}\end{cases}$

For every $\epsilon>0$ and $p>0$ we define

$$L_{\varphi,\vartheta}(\epsilon,p)=\{r\in\mathbb{N}:\varphi(\Lambda y_r-0,p)\le 1-\epsilon \text{ or } \vartheta(\Lambda y_r-0,p)\ge\epsilon\}$$
$$=\{r\in\mathbb{N}:\tfrac{p}{p+|\Lambda y_r|}\le 1-\epsilon \text{ or } \tfrac{|\Lambda y_r|}{p+|\Lambda y_r|}\ge\epsilon\}$$
$$=\left\{r\in\mathbb{N}:|\Lambda y_r|\ge\frac{\epsilon p}{1-\epsilon}>0\right\}$$
$$=\{r\in\mathbb{N}:\Lambda y_r=1\}$$
$$=\{r\in\mathbb{N}:r=m^2\}$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\varphi,\vartheta}(\epsilon,p)\right|\le\lim_{n\to\infty}\frac{\sqrt{n}}{n^\alpha}.$$

Then, $\lim_{n\to\infty}\frac{\sqrt{n}}{n^\alpha}=0$ as $n\to\infty$ if $\alpha\in(1/2,1]$

$\Rightarrow\Lambda y_r\xrightarrow{(\varphi,\vartheta)^\alpha}0(S_\Lambda^\alpha)$ i.e. $\Lambda y_r\xrightarrow{(\varphi,\vartheta)^\alpha}0$, if $\alpha\in(1/2,1]$

Since, $\varphi(\Lambda y_r,p)=\frac{p}{p+|\Lambda y_g|}=\begin{cases}\frac{p}{p+1}, & g=r^2;r\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}\le 1$

and $\vartheta(\Lambda y_g,p)=\frac{|\Lambda y_g|}{p+|\Lambda y_g|}=\begin{cases}\frac{1}{p+1}, & g=r^2;r\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}\ge 0$

Thus, the result of the theorem is established.

In the next theorem, we will show the algebraic characterization of the sequences in IFNS.

Theorem 2.9 Let $(Y,\varphi,\vartheta,*,\circ)$ be an IFNS and $0<\alpha\le 1$. Let $x=(x_r)_{r=0}^\infty$ and $y=(y_r)_{r=0}^\infty$ be two sequences in Y.

(i) If $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(x_r)=x_0$ and $b\in\mathbb{R}$, then

$$S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(bx_r)=bx_0,$$

(ii) If $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(x_r)=x_0$ and $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(y_r)=y_0$, then $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(x_r+y_r)=x_0+y_0$.

Proof: (i) Let $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(x_r)=x_0$. Then for any $\epsilon>0$ and $p>0$ we have

$$L_{\varphi,\vartheta}(\epsilon,p)=\{r\in\mathbb{N}:\varphi(\Lambda x_r-x_0,p)\le 1-\epsilon \text{ or } \vartheta(\Lambda x_r-x_0,p)\ge\epsilon\}$$

with $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|L_{\varphi,\vartheta}(\epsilon,p)\right|=0$.

If $b=0$, then the result holds. Suppose $b\ne 0$, then for $r\notin L_{\varphi,\vartheta}(\epsilon,p)$

$$\varphi(\Lambda bx_r-bx_0,p)=\varphi\left(\Lambda x_r-x_0,\frac{p}{|b|}\right)$$
$$>\varphi(\Lambda x_r-x_0,p)*\varphi\left(0,\tfrac{p}{|b|}-p\right)$$
$$>(1-\epsilon)*1=1-\epsilon$$

$\Rightarrow\varphi(\Lambda bx_r-bx_0,p)>1-\epsilon$,

and

$$\vartheta(\Lambda bx_r-bx_0,p)=\vartheta\left(\Lambda x_r-x_0,\frac{p}{|b|}\right)$$
$$<\vartheta(\Lambda x_r-x_0,p)\circ\vartheta\left(0,\frac{p}{|b|}-p\right)$$
$$<\epsilon\circ 0=\epsilon$$

$\Rightarrow\varphi(\Lambda bx_r-bx_0,p)<\epsilon$.

Thus, $\lim_{n\to\infty}\frac{1}{n^\alpha}|\{r\in\mathbb{N}:\varphi(\Lambda bx_r-bx_0,p)>1-\epsilon$ and $\vartheta(\Lambda bx_r-bx_0,p)<\epsilon\}|=1$.

i.e. $\lim_{n\to\infty}\frac{1}{n^\alpha}|\{r\in\mathbb{N}:\varphi(\Lambda bx_r-bx_0,p)\le 1-\epsilon$ or $\vartheta(\Lambda bx_r-bx_0,p)\ge\epsilon\}|=0$.

Hence, $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(bx_r)=bx_0$.

(ii) Let $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(x_r)=x_0$ and $S_\Lambda^{\varphi,\vartheta^\alpha}-\lim_{r\to\infty}(y_r)=y_0$. Then for any $\epsilon>0$ and $p>0$ we choose $w>0$ such that $(1-w)*(1-w)>1-\epsilon$ and $w\circ w<\epsilon$.

Define $L_x(w,p)=\left\{r\in\mathbb{N}:\varphi\left(\Lambda x_r-x_0,\frac{p}{2}\right)\le 1-w \text{ or } \vartheta\left(\Lambda x_r-x_0,\frac{p}{2}\right)\ge w\right\}$ with $\lim_{n\to\infty}\frac{1}{n^\alpha}|L_x(w,p)|=0$

and $L_y(w,p) = \left\{r \in \mathbb{N} : \varphi\left(\Lambda y_r - y_0, \frac{p}{2}\right) \leq 1 - w \text{ or } \vartheta\left(\Lambda y_r - y_0, \frac{p}{2}\right) \geq w\right\}$ with $\lim_{n\to\infty} \frac{1}{n^\alpha}\left|L_y(w,p)\right| = 0$.

Now, for $r \notin [L_x(w,p) \cup L_y(w,p)]$

$$\begin{aligned}\varphi(\Lambda(x_r + y_r) - (x_0 + y_0), p) &= \varphi((\Lambda x_r - x_0) + (\Lambda y_r - y_0), p) \\ &\geq \varphi\left(\Lambda x_r - x_0, \frac{p}{2}\right) + \varphi\left(\Lambda y_r - y_0, \frac{p}{2}\right) \\ &> (1-w) * (1-w) > 1 - \epsilon\end{aligned}$$

and

$$\begin{aligned}\vartheta(\Lambda(x_r + y_r) - (x_0 + y_0), p) &= \vartheta((\Lambda x_r - x_0) + (\Lambda y_r - y_0), p) \\ &\leq \vartheta\left(\Lambda x_r - x_0, \frac{p}{2}\right) \circ \vartheta\left(\Lambda y_r - y_0, \frac{p}{2}\right) \\ &< w \circ w < \epsilon\end{aligned}$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^\alpha}|\{r \in \mathbb{N} : \varphi(\Lambda(x_r + y_r) - (x_0 + y_0), p) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda(x_r + y_r) - (x_0 + y_0), p) > \epsilon\}| = 0$.

Hence, $S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{r\to\infty}(x_r + y_r) = (x_0 + y_0)$.

Theorem 2.10 Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS and $0 < \alpha \leq 1$. Then for any sequence $y = (y_r)_{r=1}^\infty$, $S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{r\to\infty}(y_r) = l$ iff there is a subset $J = \{j_1 < j_2 < j_3 \dots\dots\dots\} \subseteq \mathbb{N}$ such that $\delta_\Lambda^\alpha(J) = 1$ and $(\varphi, \vartheta) - \lim_{i\to\infty}(y_{j_i}) = l$.

Proof: Necessary Condition:

Let$S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{g\to\infty}(y_r) = l$. For any $p > 0$ and $w = 1,2,3,\dots$.

$L_{\varphi,\vartheta}(w,p)$={r ∈ ℕ : $\varphi(\Lambda y_r - l, p) > 1 - \frac{1}{w}$ and $\vartheta(\Lambda y_r - l, p) < \frac{1}{w}$} and

$L^c_{\varphi,\vartheta}(w,p)$={r ∈ ℕ : $\varphi(\Lambda y_r - l, p) \leq 1 - \frac{1}{w}$ and $\vartheta(\Lambda y_r - l, p) \geq \frac{1}{w}$}

Since, $S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{r\to\infty}(y_r) = l$.

also, $L^c_{\varphi,\vartheta}(w,p) \supset L^c_{\varphi,\vartheta}(w+1,p)$

and $\lim_{n\to\infty} \frac{1}{n^\alpha}\left|L^c_{\varphi,\vartheta}(w,p)\right| = 1$, for any $p > 0$ and $w = 1,2,3,\dots$. (2.1)

We prove, $S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{r\to\infty}(y_r) = l$ by contradiction.

Let $r \in L^c_{\varphi,\vartheta}(w,p)$. As sequence $y = (y_r)_{r=0}^\infty$ is not statistically Λ -convergent.

$\exists\, \beta > 0$ and $r_0 \in \mathbb{Z}^+$ such that $\varphi(\Lambda y_r - l, p) \leq 1 - \beta$ or $\vartheta(\Lambda y_r - l, p) \geq \beta \;\; \forall\, r \geq r_0$

i.e. $\varphi(\Lambda y_r - l, p) > 1 - \beta$and $\vartheta(\Lambda y_r - l, t) < \beta \;\;\forall\, r < r_0$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^\alpha}|\{r \in \mathbb{N} : \varphi(\Lambda y_r - l, p) > 1 - \beta \text{ or } \vartheta(\Lambda y_r - l, p) \leq \beta\}| = 0.$$

Since, $\beta > \frac{1}{w}$, we get $\lim_{n\to\infty} \frac{1}{n^\alpha}\left|L^c_{\varphi,\vartheta}(w,p)\right| = 0$, which is a contradiction to the equation (2.1).

Sufficient Condition:

Suppose $\exists$ a set $= \{j_1 < j_2 < j_3 \dots\dots\dots\} \subseteq \mathbb{N}$ such that $\delta(J) =$ and $S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{i\to\infty}(y_{j_i}) = l$.

i.e. $\exists\, r \in \mathbb{N}$ such that $\beta > 0$ and $p > 0$.

$\varphi(\Lambda y_r - l, p) > 1 - \beta$ or $\vartheta(\Lambda y_r - l, p) \leq \beta$

Now,

$$\begin{aligned}L_{\varphi,\vartheta}(w,p) = \{r \in \mathbb{N} : \varphi(\Lambda y_g - l, p) &\leq 1 - \frac{1}{w} \text{ or } \vartheta(\Lambda y_r - l, p) \geq \frac{1}{w}\} \\ &\subseteq \mathbb{N} - \{j_{r+1}, j_{r+2} \dots\dots\}\end{aligned}$$

$\lim_{n\to\infty} \frac{1}{n^\alpha}\left|L_{\varphi,\vartheta}(w,p)\right| = 1 - 1 = 0$.

$S_\Lambda^{\varphi,\vartheta^\alpha} - \lim_{i\to\infty}(y_{j_i}) = l$.

Hence proved.

Definition 2.11 Let $(Y, \varphi, \vartheta, *, \circ)$ be an IFNS and $y = (y_r)_{r=0}^\infty$ be any sequence. Then it is said to be statistically Λ -Cauchy sequence of order $\alpha(0 < \alpha \leq 1)$ with respect to (φ, ϑ), if for $w > 0$ and $p > 0 \;\exists\; N = N(w)$ such that

$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{r \in \mathbb{N} : \varphi(\Lambda y_r - y_N, p) \leq 1 - w \text{ or } \vartheta(\Lambda y_r - y_N, p) \geq w\}| = 0$.

In the next theorem, we will prove that statistically Λ –convergent is statistically Λ-Cauchy sequence w.r.t. (φ, ϑ).

Theorem 2.12 Let$(Y, \varphi, \vartheta, *, \circ)$ be an IFNS and $y = (y_r)_{r=0}^\infty$ be any sequence. Then it is statistically Λ-Cauchy sequence of order $\alpha(0 < \alpha \leq 1)$ with respect to (φ, ϑ), if and only if it is $S_\Lambda^{\varphi,\vartheta^\alpha}$-convergent.

Proof: Let $S_\Lambda^{\varphi,\vartheta^\alpha} - \lim(y_r) = l$. Then for any $\epsilon > 0$ we choose $w > 0$ such that $(1-w) * (1-w) > 1 - \epsilon$ and $w \circ w < \epsilon$.

Now for any $p > 0$, we have

$L_{\varphi,\vartheta}(w,p) = \{r \in \mathbb{N} : \varphi\left(\Lambda y_r - l, \frac{p}{2}\right) \leq 1 - w \text{ or } \vartheta\left(\Lambda y_r - l, \frac{p}{2}\right) \geq w\} = 0$, (2.2)

$L^c_{\varphi,\vartheta}(w,p) = \{r \in \mathbb{N} : \varphi\left(\Lambda y_r - l, \frac{p}{2}\right) > 1 - w \text{ and } \vartheta\left(\Lambda y_r - l, \frac{p}{2}\right) < w\} = 1$.

Let $d \in L^c_{\varphi,\vartheta}(w,p)$. Then $\varphi\left(\Lambda y_d - l, \frac{p}{2}\right) > 1 - w$ and $\vartheta\left(\Lambda y_d - l, \frac{p}{2}\right) < w$.

Now, let $C_{\varphi,\vartheta}(w,p) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_d, p) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \Lambda y_d, p)\} \geq \epsilon$.

We need to show that $C_{\varphi,\vartheta}(w,p) \subset L_{\varphi,\vartheta}(w,p)$.

Let$r \in C_{\varphi,\vartheta}(w,p) - L_{\varphi,\vartheta}(w,p)$.

Now we have, $\varphi(\Lambda y_r - \Lambda y_d, p) \leq 1 - \epsilon$ and $\varphi\left(\Lambda y_r - l, \frac{p}{2}\right) > 1 - w$.

In particular, $\varphi\left(\Lambda y_d - l, \frac{p}{2}\right) > 1 - w$, then

$1 - \epsilon \geq \varphi(\Lambda y_r - \Lambda y_d, p) \geq \varphi\left(\Lambda y_r - l, \frac{p}{2}\right) * \varphi\left(\Lambda y_d - l, \frac{p}{2}\right) > (1-w) * (1-w) > 1 - \epsilon$ which is a contradiction.

On the other hand, $\vartheta(\Lambda y_r - \Lambda y_d, p) \geq \epsilon$ and $\vartheta\left(\Lambda y_r - l, \frac{p}{2}\right) < w$.

In particular, $\vartheta\left(\Lambda y_d - l, \frac{p}{2}\right) < w$.

Then,

$$\begin{aligned}\epsilon \leq \vartheta(\Lambda y_r - \Lambda y_d, p) \leq \vartheta\left(\Lambda y_r - l, \frac{p}{2}\right) \circ \vartheta\left(\Lambda y_d - l, \frac{p}{2}\right) \\ < w \circ w < \epsilon.\end{aligned}$$

It contradicts again.

Hence, $C_{\varphi,\vartheta}(w,p) \subset L_{\varphi,\vartheta}(w,p)$.

Therefore, by equation (2.2)

$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{r \in \mathbb{N} : \varphi(\Lambda y_r - y_d, p) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - y_d, p) \geq \epsilon\}| = 0$.

Hence, $y = (y_r)_{r=0}^\infty$ is $S_\Lambda^{\varphi,\vartheta^\alpha}$-Cauchy with respect to (φ, ϑ).

Conversely,

Let $y = (y_r)_{r=0}^\infty$ be $S_\Lambda^{\varphi,\vartheta^\alpha}$-Cauchy with respect to (φ, ϑ) but not $S_\Lambda^{\varphi,\vartheta^\alpha}$-convergent.

$\lim_{n\to\infty} \frac{1}{n^\alpha}\left|L_{\varphi,\vartheta}(w,p)\right| = 0$ where

$$L_{\varphi,\vartheta}(w,p) = \{r \in \mathbb{N}: \varphi(\Lambda y_r - \Lambda y_d, p) \leq 1 - w \text{ or } \vartheta(\Lambda y_r - \Lambda y_d, p) \geq w\} = 0.$$

Choose $\epsilon > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > (1 - w)$ and $\epsilon \circ \epsilon < w$.

Now, $\varphi(\Lambda y_r - \Lambda y_d, p) \geq \varphi\left(\Lambda y_r - l, \frac{p}{2}\right) * \varphi\left(\Lambda y_d - l, \frac{p}{2}\right)$
$> (1-\epsilon) * (1-\epsilon)$
$> 1 - w$

and $\vartheta(\Lambda y_r - \Lambda y_d, p) \leq \vartheta\left(\Lambda y_r - l, \frac{p}{2}\right) \circ \vartheta\left(\Lambda y_d - l, \frac{p}{2}\right)$
$< \epsilon \circ \epsilon < w.$

Since, $y = (y_r)_{r=0}^{\infty}$ is not $S_\Lambda^{\varphi,\vartheta^\alpha}$-convergent.

Therefore, $\lim_{n\to\infty} \frac{1}{n^\alpha} \left|L_{\varphi,\vartheta}^c(w,p)\right| = 1 - 1 = 0$ i.e.

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^\alpha} \left|L_{\varphi,\vartheta}(w,p)\right| = 1,$

which is a contradiction as $y = (y_r)_{r=0}^{\infty}$ is $S_\Lambda^{\varphi,\vartheta^\alpha}$-Cauchy.

Hence, $y = (y_r)_{r=0}^{\infty}$ is $S_\Lambda^{\varphi,\vartheta^\alpha}$-convergent with respect to (φ, ϑ).

III. CONCLUSION

The paper concludes the generalization of the Statistical convergence with the help of $\Lambda-$convergence over Intuitionistic fuzzy norm space and also present some fundamental properties related to this concept. Also, the paper gives more generalized results as compared to classical convergence methods.

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